§4.6 Operator Methods

Time ordering

Recall that the derivation of the path integral expression involved the crucial step:

$$\langle q';\tau+d\tau|q;\tau\rangle = \langle q';\tau|e^{-iHd\tau}|q;\tau\rangle \quad (1)$$

Hamiltonian H is function $H(Q,P)$

$$\longrightarrow \quad H(Q,P) = e^{iHt} H(Q,P) e^{-iHt}$$
$$= H(Q(t), P(t))$$

adopt a standard form, where all Qs appear to the <u>left</u> of all P's.

<u>example</u>: given term $P_a Q_b P_c \subset H$,
rewrite as $Q_b P_a P_c - i\delta_{ab} P_c$

$\longrightarrow$ substituting into (1), we see that $Q_a(t)$s can be replaced with q'_a

Then, proceeding as in §1,

$$\langle q'; \tau + d\tau | q; \tau\rangle$$

$$= \int \prod_a \frac{dp_a}{2\pi} \langle q'; \tau | \exp\left(-i H(Q(\tau), P(\tau)) d\tau\right) | p; \tau\rangle \times \langle p; \tau | q; \tau\rangle$$

$$= \int \prod_a \frac{dp_a}{2\pi} \exp\left[-i H(q', p)\, d\tau + i \sum_a (q'_a - q_a) p_a\right] \qquad (2)$$

with p_a integrated from $-\infty$ to ∞.

For a finite time interval, to calculate $\langle q'; t' | q; t\rangle$ with $t < t'$, we write

$$\tau_{k+1} - \tau_k = d\tau = \frac{t' - t}{N+1}$$

and

$$\langle q'; t' | q; t\rangle$$

$$\overset{!}{=} \int dq_1 \cdots dq_N \langle q'; t' | q_N; \tau_N\rangle \langle q_N; \tau_N | q_{N-1}; \tau_{N-1}\rangle \cdots \langle q_1; \tau_1 | q; t\rangle$$

$\longrightarrow$ inserting eq. (2) gives

$$\overset{!}{=} \int \left[\prod_{k=1}^{N} \prod_a dq_{k,a}\right] \left[\prod_{k=0}^{N} \prod_a dp_{k,a}/2\pi\right]$$

$$\times \exp\left[i \sum_{k=1}^{N+1} \left(\sum_a (q_{k,a} - q_{k-1,a})(p_{k-1,a} - H(q_k, p_{k-1}) d\tau \right) \right]$$

where $\quad q_0 := q \, , \quad q_{N+1} := q'$

Taking the limit $N \to \infty$, gives

$$\langle q'; t' | q; t \rangle = \int_{\substack{q_a(t) = q_a \\ q_a(t') = q_a'}} \prod_{\tau, a} dq_a(\tau) \prod_{\tau, b} \frac{dp_b(\tau)}{2\pi}$$

$$\times \exp\left[i \int_t^{t'} d\tau \left(\sum_a \dot{q}_a(\tau) p_a(\tau) - H(q(\tau), p(\tau)) \right) \right] \quad (3)$$

integrating out p then gives the path integral with the Legendre trf. replaced by the Lagrangian L

Operator insertions

Consider now general operators $O(P(\tau), Q(\tau))$, where we order all Ps in such operators to the left of all Qs.

$\longrightarrow$ inserting into (1), gives

$$\langle q'; \tau + d\tau | \mathcal{O}(P(\tau), Q(\tau) | q; \tau \rangle$$

$$= \int \prod_a \frac{dp_a}{2\pi} \langle q'; \tau | \exp\left(-i H(Q(\tau), P(\tau)) d\tau\right) | p; \tau \rangle$$

$$\times \langle p; \tau | \mathcal{O}(P(\tau), Q(\tau) | q; \tau \rangle$$

$$= \int \prod_a \frac{dp_a}{2\pi} \exp\left[-i H(q', p) d\tau + i \sum_a (q'_a - q_a) p_a\right] \mathcal{O}(p, q) \quad (4)$$

$\rightarrow$ this allows us to calculate the matrix element of a product $\mathcal{O}_A(P(t_A), Q(t_A)) \mathcal{O}_B(P(t_B), Q(t_B)) \cdots$ of operators with $t_A > t_B > \cdots$, by inserting the $\mathcal{O}$-operators between the appropriate states in eq. (3) and use (4)

For example, if t_A falls between τ_k and τ_{k+1}, then insert $\mathcal{O}_A(P(t_A), Q(t_A))$ between $\langle q_{k+1}; \tau_{k+1}|$ and $|q_k; \tau_k\rangle$

$\rightarrow$ this is only possible for $t_A > t_B > \cdots$

Following same steps as before, finally gives

$$\langle q', t' | \mathcal{O}_A(P(t_A), Q(t_A)) \mathcal{O}_B(P(t_B), Q(t_B)) \cdots | q, t\rangle$$

$$= \int_{\substack{q_a(t) = q_a \\ q_a(t') = q'_a}} \prod_{\tau, a} dq_a(\tau) \prod_{\tau, b} \frac{dp_b(\tau)}{2\pi} \mathcal{O}_A(p(t_A), q(t_A)) \mathcal{O}_B(p(t_B), q(t_B)) \cdots \times \exp\left[i \int_t^{t'} d\tau \left(\sum_a \dot{q}_a(\tau) p_a(\tau) - H(q(\tau), p(\tau)) \right) \right] \quad (5)$$

→ result only valid if times ordered as

$$t' > t_A > t_B > \cdots > t$$

For arbitrary order, the correct expression is

$$\langle q', t' | T\left[\mathcal{O}_A(P(t_A), Q(t_A)) \mathcal{O}_B(P(t_B), Q(t_B)) \cdots \right] | q, t\rangle$$

$$= \int_{\substack{q_a(t) = q_a \\ q_a(t') = q'_a}} \prod_{\tau, a} dq_a(\tau) \prod_{\tau, b} \frac{dp_b(\tau)}{2\pi} \mathcal{O}_A(p(t_A), q(t_A)) \mathcal{O}_B(p(t_B), q(t_B)) \cdots \times \exp\left[i \int_t^{t'} d\tau \left(\sum_a \dot{q}_a(\tau) p_a(\tau) - H(q(\tau), p(\tau)) \right) \right] \quad (6)$$

In QFT we are interested in transition amplitudes between "in" and "out" states $|\Psi_\alpha, \text{in}\rangle$, $|\Psi_\beta, \text{out}\rangle$, where Ψ_α, Ψ_β denote set of particles characterized by various momenta, spin z-component, and species.

$\longrightarrow$ to calculate matrix element, we need to multiply (6) by $\langle \Psi_\beta, \text{out} | q', t' \rangle$ and $\langle q, t | \Psi_\alpha, \text{in} \rangle$ and perform integral over q', q

$$i\int_t^{t'} d\tau \left(\sum_a \dot{q}_a(\tau) p_a(\tau) - H(q(\tau), p(\tau)) \right) \Big]$$

$$\longrightarrow \langle \Psi_\beta, \text{out} | T\left[O_A(P(t_A), Q(t_A)) O_B(P(t_B), Q(t_B)) \cdots \right] | \Psi_\alpha, \text{in} \rangle$$

$$= \int \prod_{\tau, x, m} dq_m(\vec{x}, \tau) \prod_{\tau, x, m} \left(\frac{dp_m(\vec{x}, \tau)}{2\pi} \right)$$

$$\times O_A(p(t_A), q(t_A)) O_B(p(t_B), q(t_B)) \cdots$$

$$\times \exp\left[i \int_{-\infty}^{\infty} d\tau \left(\int d^3x \sum_m \dot{q}_m(\vec{x}, \tau) p_m(\vec{x}, \tau) - H(q(\tau), p(\tau)) \right) \right]$$

$$\times \langle \Psi_\beta, \text{out} | q(+\infty); +\infty \rangle \langle q(-\infty); -\infty | \Psi_\alpha, \text{in} \rangle$$

"S-matrix"

Generalized Ward Identity

Consider field theory invariant under global gauge trfs. $\Phi_\ell \mapsto \exp(i q_\ell \alpha)\Phi_\ell$

$\longrightarrow$ conserved current

$$J^\mu = -i \sum_\ell \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\ell)} q_\ell \Phi_\ell \,, \quad \partial_\mu J^\mu = 0$$

$$\longrightarrow \quad i \frac{d}{dt} Q = [Q, H] = 0 \,, \quad Q := \int d^3x \, J^0$$

We have $[\underset{\text{translation gen.}}{\vec{P}}, Q] = 0$, $[\underset{\text{Lorentz gen.}}{J^{\mu\nu}}, Q] = 0$

Vacuum $|0\rangle$ is Lorentz-invariant state of zero energy and momentum

$$\longrightarrow \quad Q|0\rangle \sim |0\rangle$$

Proportionality constant is fixed by noting $\langle 0 | J_\mu | 0 \rangle = 0$ by Lorentz inv.

$$\longrightarrow \quad Q|0\rangle = 0$$

Also $Q|\Psi_{\vec{p},\sigma,n}\rangle = \underset{\text{electric charge}}{q_{(n)}} |\Psi_{\vec{p},\sigma,n}\rangle$

We have

$$[J^0(\vec{x},t), \Phi_\ell(\vec{y},t)] = -q_\ell \Phi_\ell(\vec{y},t)\, \delta^{(3)}(\vec{x}-\vec{y})$$

$$\longrightarrow \quad [Q, \Phi_\ell(y)] = -q_\ell \Phi_\ell(y)$$

Same is true for any local function $F(y)$ of fields and field derivatives

$$[Q, F(y)] = -q_F F(y)$$

sum of all q_ℓ's for all fields and field ders.

$$\longrightarrow \quad 0 = \langle 0| Q\, F(y)\, |\Psi_{\vec{p},\sigma,n}\rangle$$

$$= \langle 0|\, [Q, F(y)]\, \Psi_{\vec{p},\sigma,n}|0\rangle$$

$$+ \langle 0|\, F(y)\, \underbrace{Q\, |\Psi_{\vec{p},\sigma,n}\rangle}_{= q_{(n)} |\Psi_{\vec{p},\sigma,n}\rangle}$$

$$= \langle 0|\, F(y)\, |\Psi_{\vec{p},\sigma,n}\rangle (q_F - q_{(n)})$$

$\longrightarrow \quad q_{(n)} = q_F$ as long as

$$\langle 0| F(y) |\Psi_{\vec{p},\sigma,n}\rangle \neq 0$$